Quantum motion in superposition of Aharonov-Bohm with some additional electromagnetic fields

V.G. Bagrov*, D.M. Gitman†, and A.D. Levin‡

January 18, 2013

Abstract

The structure of additional electromagnetic fields to the Aharonov-Bohm field, for which the Schrödinger, Klein-Gordon, and Dirac equations can be solved exactly are described and the corresponding exact solutions are found. It is demonstrated that aside from the known cases (a constant and uniform magnetic field that is parallel to the Aharonov-Bohm solenoid, a static spherically symmetrical electric field, and the field of a magnetic monopole), there are broad classes of additional fields. Among these new additional fields we have physically interesting electric fields acting during a finite time, or localized in a restricted region of space. There are additional time-dependent uniform and isotropic electric fields that allow exact solutions of the Schrodinger equation. In the relativistic case there are additional electric fields propagating along the Aharonov-Bohm solenoid with arbitrary electric pulse shape.

Introduction

The Aharonov-Bohm (AB) effect plays an important role in quantum theory revealing a peculiar status of electromagnetic potentials in the theory. This effect was discussed in [1] when studying the scattering of a non-relativistic charged spinless particle by an infinitely long and infinitely thin magnetic field of a solenoid (the AB field in what follows) of finite magnetic flux (a similar effect was discussed earlier by Ehrenberg and Siday [2]). The Schrödinger equation with such a field was exactly solved and it was found that a particle wave function vanishes on the solenoid line. Although the particle does not penetrate the solenoid, while the magnetic field vanishes outside of it, the partial scattering phases are proportional to the magnetic flux (modulo a flux quantum) [3]. A nontrivial particle scattering by such a field was interpreted as a capability of a quantum particle to "feel" an electromagnetic field vector potential because the AB field vector potential does not vanish outside of the solenoid[1]. Solutions of the Dirac equations with AB field were found and studied in detail, see e.g. [4, 5, 6, 7, 9, 10, 11].

A splitting of Landau levels in a superposition of the AB field and a parallel uniform magnetic field (we call such a superposition the magnetic-solenoid field-MSF) gives an example

*Department of Physics, Tomsk State University, 634050, Tomsk, Russia. Tomsk Institute of High Current Electronics, SB RAS, 634034 Tomsk, Russia; e-mail: bagrov@phys.tsu.ru

†Institute of Physics, University of Sao Paulo, Brazil; e-mail: gitman@dfn.if.usp.br

‡Institute of Physics, University of Sao Paulo, Brazil; e-mail: alevin@if.usp.br

[1]It should be mentioned that in the relativistic case (Dirac equation with AB field) some of wave functions from a complete set of solutions do not vanish on the solenoid line.

of the AB effect for bound states. Solutions of the non-relativistic Schrödinger equation with MSF were first studied in [12]. Solutions of the Klein-Gordon and Dirac equations with MSF were first obtained in [13] and then studied in detail in [14, 15, 16, 17, 18]. It is important to stress that in contrast to the pure AB field case, where particles effectively interact with the solenoid for a finite short time, the particles moving in MSF interact with the solenoid permanently. This opens more possibilities to study such an interaction in a number of corresponding real physical situations. For example, using these solutions the AB effect in cyclotron and synchrotron radiations was calculated in [19, 20, 21]. Recently interest in such a superposition has been renewed in connection with planar physics problems and the quantum Hall effect [8, 14, 22]. The example of the MSF stresses the importance of studying quantum motion in superposition of the AB field with some additional electromagnetic fields. It should be noted that exact solutions of the Schrödinger, Klein-Gordon and Dirac equations with the AB field in combination with the Coulomb field and the magnetic monopole field were studied in [23, 24, 25, 26, 27, 13]. Exact solutions of the above mentioned equations with the AB field in combination with some other electromagnetic fields were presented in [28, 13, 29].

The aim of the present work is to find the structure of the additional electromagnetic fields, for which the Schrödinger, Klein-Gordon, and Dirac equations can be solved exactly (in what follows, we call such fields exactly solvable additional fields), and to describe the corresponding exact solutions.

1 Aharonov-Bohm field and its combination with additional electromagnetic fields

In sections 1 and 2, we use Lorentz coordinates $x^{\nu} = (x^0 = ct, x, y, z)$, and cylindrical coordinates r, φ in the x, y-plane ($x = r\cos\varphi,\ y = r\sin\varphi$).

The Aharonov-Bohm field is a magnetic field $\mathbf{B}$ of an infinitesimally thin solenoid with magnetic flux Φ,

$$B_x = B_y = 0,\ B_z = \Phi\,\delta(x)\,\delta(y) = \frac{\Phi}{\pi r}\delta(r),\ r^2 = x^2 + y^2,\ \Phi = \text{const}. \tag{1}$$

It can be described by the potentials $A_{\nu}^{(0)}$ of the form

$$\begin{aligned} & A_{\nu}^{(0)} = \left(A_0^{(0)}, -\mathbf{A}^{(0)}\right),\quad \mathbf{A}^{(0)} = \left(A_x^{(0)},\ A_y^{(0)},\ A_z^{(0)}\right), \\ & A_0^{(0)} = A_z^{(0)} = 0,\quad A_x^{(0)} = -\frac{\Phi}{2\pi}\frac{y}{r^2} = \frac{\Phi}{2\pi}\frac{\partial\varphi}{\partial x},\quad A_y^{(0)} = \frac{\Phi}{2\pi}\frac{x}{r^2} = \frac{\Phi}{2\pi}\frac{\partial\varphi}{\partial y}, \\ & \mathbf{A}^{(0)} = \frac{\Phi}{2\pi r}\mathbf{e}_{\varphi},\quad \mathbf{e}_{\varphi} = -\mathbf{i}\sin\varphi + \mathbf{j}\cos\varphi\,. \end{aligned} \tag{2}$$

We denote by $\mathbf{i}, \mathbf{j}, \mathbf{k}$ three unit orthogonal vectors along the Cartesian axis x, y, z, whereas $\mathbf{e}_{\varphi}$ and $\mathbf{e}_r$ are unit orthogonal vectors of the cylindrical coordinate φ, r system.

An important characteristic of the AB field is the mantissa μ of the magnetic flux Φ, which is defined as follows

$$\Phi = -\left(e/|e|\right)(l_0 + \mu)\Phi_0,\quad 0 \leqslant \mu < 1, \tag{3}$$

where l_0 is an integer, $\Phi_0 = 2\pi c\hbar/|e|$ is the Dirac quantum of the magnetic flux, and e the algebraic particle charge. Then

$$\frac{e}{c}A_{\nu}^{(0)} = \left(0, -\frac{e}{c}\mathbf{A}^{(0)}\right) = \hbar(l_0 + \mu)\partial_{\nu}\varphi,\quad \frac{e}{c}\mathbf{A}^{(0)} = -\hbar\frac{l_0 + \mu}{r}\mathbf{e}_{\varphi}. \tag{4}$$

Consider a linear combination of electromagnetic potentials of the AB field with an additional electromagnetic field. Potentials $\mathcal{A}_\nu$ of such a composite field, we represent as

$$\mathcal{A}_\nu = A_\nu + A_\nu^{(0)} = \left(A_0, -\mathbf{A} - \mathbf{A}^{(0)}\right). \tag{5}$$

The Schrödinger, Klein-Gordon, and Dirac equations with such a composed field have the form

$$\mathcal{S}\Psi_S = 0, \quad \mathcal{S} = cP_0 - \frac{\mathbf{P}^2}{2m_0}; \tag{6}$$
$$\mathcal{K}\Psi_K = 0, \quad \mathcal{K} = P^\nu P_\nu - m_0^2c^2 = P_0^2 - \mathbf{P}^2 - m_0^2c^2; \tag{7}$$
$$\mathcal{D}\Psi_D = 0, \quad \mathcal{D} = \gamma^\nu P_\nu - m_0c = \gamma^0 P_0 - (\boldsymbol{\gamma}\mathbf{P}) - m_0c. \tag{8}$$

Here m_0 is the particle rest mass, γ^ν are Dirac gamma matrices (we use here standard representation for them), and covariant components of the kinetic momentum operator P_ν are

$$\begin{aligned} P_\nu &= (P_0, -\mathbf{P}) = p_\nu - \frac{e}{c}\mathcal{A}_\nu, \quad P_0 = p_0 - \frac{e}{c}\mathcal{A}_0, \quad \mathbf{P} = \mathbf{p} - \frac{e}{c}\boldsymbol{\mathcal{A}}; \\ p_\nu &= (p_0, -\mathbf{p}) = i\hbar\partial_\nu, \quad p_0 = i\hbar\partial_0, \quad \mathbf{p} = -i\hbar\nabla, \end{aligned} \tag{9}$$

where p_ν are covariant components of the generalized momentum operator.

In what follows, we consider only the nontrivial AB field with nonzero mantissa μ. As follows from (2), in such a case cylindrical and spherical coordinates are physically preferable. Namely, in these coordinates equations (2) with the AB field and exactly solvable additional fields allow separation of the variables and exact solutions can be obtained.

2 Structure of additional electromagnetic fields. Cylindrical coordinates

Let us write potentials A_ν of the additional field in the form

$$\frac{e}{c\hbar}A_0 = f_0(r,\, z,\, x_0), \quad \frac{e}{c\hbar}\mathbf{A} = \frac{f_2(r)}{r}\mathbf{e}_\varphi - f_1(r,\, z,\, x_0)\mathbf{k}, \tag{10}$$

where f_k $(k = 0, 1, 2)$, some arbitrary functions of the indicated arguments. Thus, potentials of the AB field (2) can be considered as a particular case of (10) for $f_0(r\, z, x_0) = f_1(r, z, x_0) = 0$, and $f_2(r) = \Phi/2\pi = \text{const}$.

Electric $\mathbf{E}$ and magnetic $\mathbf{H}$ fields that correspond to potentials (10) are

$$\begin{aligned} \frac{e}{c\hbar}\mathbf{E} &= -\partial_r f_0(r,\, z,\, x_0)\mathbf{e}_r + [\partial_0 f_1(r,\, z,\, x_0) - \partial_z f_0(r,\, z,\, x_0)]\mathbf{k}, \\ \frac{e}{c\hbar}\mathbf{H} &= \partial_r f_1(r,\, z,\, x_0)\mathbf{e}_\varphi + \frac{f_2'(r)}{r}\mathbf{k}; \quad \mathbf{e}_r = \mathbf{i}\cos\varphi + \mathbf{j}\sin\varphi. \end{aligned} \tag{11}$$

In such fields, the Schrödinger and Klein-Gordon equations have an integral of motion

$$L_z = [\mathbf{r} \times \mathbf{p}]_z = -i\hbar\,\partial_\varphi \tag{12}$$

and the Dirac equation has an integral of motion of the form

$$J_z = L_z + \frac{\hbar}{2}\Sigma_3 = -i\hbar\,\partial_\varphi + \frac{\hbar}{2}\Sigma_3. \tag{13}$$

Let us look for solutions of the Schrödinger, Klein-Gordon, and Dirac equations with the potentials (10), solutions that are eigenfunctions of the operators (12) and (13) the eigenvalues

$$L_z \to \hbar(l - l_0);\ J_z \to \hbar(l - l_0 - \frac{1}{2}),\ l \in \mathbb{Z},$$

respectively.

For the Schrödinger (S) and Klein-Gordon (K) equations such solutions have the form

$$\Psi_{S,K}(x) = \exp(iQ)\psi_{S,K}(r,z,x_0), \quad Q = (l - l_0)\varphi, \tag{14}$$

where the functions $\psi_{S,K}(r,z,x_0)$ obey the equations

$$\left\{2m\pi_0 + \partial_r^2 + \frac{1}{r}\partial_r - \frac{[f_2(r) - l - \mu]^2}{r^2} - \pi_3^2\right\}\psi_S(r,z,x_0) = 0, \tag{15}$$

$$\left\{\pi_0^2 + \partial_r^2 + \frac{1}{r}\partial_r - \frac{[f_2(r) - l - \mu]^2}{r^2} - \pi_3^2 - m^2\right\}\psi_K(r,z,x_0) = 0. \tag{16}$$

Here

$$\pi_0 = i\partial_0 - f_0(r,z,x_0), \quad \pi_3 = i\partial_z - f_1(r,z,x_0), \quad m = \frac{m_0 c}{\hbar}. \tag{17}$$

For the Dirac equation (D), such solutions have the form

$$\Psi_{D;l}(x^\nu) = e^{iQ}\begin{pmatrix} e^{-i\varphi}\psi_1(r,\ z,\ x_0) \\ i\psi_2(r,\ z,\ x_0) \\ e^{-i\varphi}\psi_3(r,\ z,\ x_0) \\ i\psi_4(r,\ z,\ x_0) \end{pmatrix}, \tag{18}$$

where the functions $\psi_s(r,\ z,\ x_0),\ \ s = 1,\ 2,\ 3,\ 4$, obey the following set of equations

$$\tilde{D}\tilde{\Phi} = 0,\ \ \tilde{\Phi} = \begin{pmatrix} \psi_1(r,\ z,\ x_0) \\ \psi_2(r,\ z,\ x_0) \\ \psi_3(r,\ z,\ x_0) \\ \psi_4(r,\ z,\ x_0) \end{pmatrix}, \tag{19}$$

with the matrix operator $\tilde{D}$ having the form

$$\overline{D} = \rho_3\pi_0 + i\rho_2\Sigma_3\pi_3 + \rho_2\Sigma_2\left(\partial_r + \frac{1}{2r}\right) + i\rho_2\Sigma_1\frac{f_2(r) - l - \mu + 1/2}{r} - mI. \tag{20}$$

Here $\rho_k,\ \Sigma_k,\ k = 1,\ 2,\ 3$, are Dirac matrices in the standard representation, and $\mathbb{I}$ a unit 4×4 matrix.

Exact solutions of eqs. (15), (16), and (19) are known only for two types of function $f_2(r)$:

$$a)\ f_2(r) = \gamma r;\quad b)\ f_2(r) = \gamma r^2, \quad \gamma = \text{const}. \tag{21}$$

3 Klein-Gordon and Dirac equations

Exact solutions of eqs. (16) and (19) are known in the two cases considered below.

3.1 Case I:

In this case the functions f_0 and f_1 depend on r only ($f_{0,1} = f_{0,1}(r)$) and are linearly dependent.

Let us consider solutions of eqs. (16) and (19) that are eigenfunctions for both operators $i\partial_0$ and $i\partial_z$ with the eigenvalues k_0 and k_3 respectively. Such solutions have the form (14) and (18) with

$$Q = (l-l_0)\varphi - k_0 x_0 - k_3 z, \quad \pi_0 = k_0 - f_0(r), \quad \pi_3 = k_3 - f_1(r); \quad \psi_K = \psi_K(r), \quad \psi_s = \psi_s(r). \tag{22}$$

Equations for the functions $\psi_K(r)$ and $\psi_s(r)$ hold the form (16) and (19) with the natural substitution $\partial_r \to d/dr$.

Let us suppose that the functions $f_0(r)$ and $f_1(r)$ are linearly dependent. In such a case, with the help of a Lorentz transformation, which does not changes the function $f_2(r)$, one can reduce the problem to the following non-equivalent subcases:

$$\begin{aligned} 1)\ f_0(r) &= f(r) \neq 0, \quad f_1(r) = 0; \\ 2)\ f_0(r) &= 0, \quad f_1(r) = f(r) \neq 0; \\ 3)\ f_0(r) &= \epsilon f_1(r) = f(r), \quad \epsilon = \pm 1. \end{aligned}$$

They are considered separately below.

3.1.1 Subcase 1

This subcase is characterized by the conditions

$$\begin{aligned} & f_0(r) = f(r) \neq 0, \quad f_1(r) = 0; \\ & \frac{e}{c\hbar}\mathbf{E} = -f'(r)\mathbf{e}_r, \quad \frac{e}{c\hbar}\mathbf{H} = r^{-1} f_2'(r)\mathbf{k}\ . \end{aligned}$$

For such fields, the Dirac equation admits a spin integral of motion [30] $T_1 = m\rho_3\Sigma_3 - k_3\rho_1$. We consider solutions that are its eigenfunctions,

$$T_1 \Psi_{D;l}(x^\nu) = \zeta\lambda_1 \Psi_{D;l}(x^\nu), \quad \zeta = \pm 1, \quad \lambda_1 = \sqrt{m^2 + k_3^2}. \tag{23}$$

It follows from (23) with account taken of (18) that

$$\begin{aligned} & \psi_1(r) = a\varphi_1(r), \quad \psi_2(r) = -b\varphi_2(r), \quad \psi_3(r) = b\varphi_1(r), \quad \psi_4(r) = -a\varphi_2(r); \\ & a = \lambda_1 + m + k_3 + \zeta(\lambda_1 + m - k_3), \quad b = \lambda_1 + m - k_3 - \zeta(\lambda_1 + m + k_3). \end{aligned}$$

Equations for functions $\varphi_1(r)$ and $\varphi_2(r)$ follow from (19),

$$\begin{aligned} & [k_0 - f(r) - \zeta\lambda_1]\varphi_1(r) - \left[\frac{f_2(r) - l - \mu}{r} - \frac{d}{dr}\right]\varphi_2(r) = 0, \\ & \left[\frac{f_2(r) - l - \mu + 1}{r} + \frac{d}{dr}\right]\varphi_1(r) - [k_0 - f(r) + \zeta\lambda_1]\varphi_2(r) = 0. \end{aligned} \tag{24}$$

Then the Klein-Gordon equation (16) takes the form

$$\left\{\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + [k_0 - f(r)]^2 - \frac{[f_2(r) - l - \mu]^2}{r^2} - k_3^2 - m^2\right\}\psi_K(r) = 0. \tag{25}$$

Solutions of the set (24) and eq. (25) (for nonzero $f(r)$) are known only for

$$f_2(r) = \gamma r, \quad f(r) = \frac{\alpha}{r}; \quad \alpha \neq 0, \quad \gamma = \text{const.}$$

These solutions have the form

$$\begin{aligned} \varphi_1(r) &= \sqrt{q_{1(-)}\omega_1(\zeta)}\, v_1(x) + \sqrt{q_{1(+)}\omega_1(-\zeta)}\, v_2(x), \\ \varphi_2(r) &= \sqrt{q_{1(+)}\omega_1(\zeta)}\, v_1(x) + \sqrt{q_{1(-)}\omega_1(-\zeta)}\, v_2(x), \\ \psi_K(r) &= v_0(x), \end{aligned} \tag{26}$$

where

$$q_{1(\pm)} = l + \mu - \frac{1}{2} \pm \sqrt{\left(l + \mu - \frac{1}{2}\right)^2 - \alpha^2}, \quad E = \sqrt{m^2 + k_3^2 + \gamma^2 - k_0^2},$$

$$\omega_1(\zeta) = \gamma\alpha - k_0\left(l + \mu - \frac{1}{2}\right) + \zeta\lambda_1\sqrt{\left(l + \mu - \frac{1}{2}\right)^2 - \alpha^2}, \quad x = 2rE.$$

The functions $v_s(x)$, $(s = 0,\, 1,\, 2)$ have similar structure,

$$v_s(x) = AI_{p_s,\,n_s}(x) + BI_{n_s,\,p_s}(x), \tag{27}$$

where A and B are arbitrary constants, and $I_{p,\,n}(x)$ Laguerre functions, related to the confluent hypergeometric function $\Phi(\alpha, \gamma; x)$ by the relation (see [31], pp. 1072 -1073) as

$$I_{p,\,n}(x) = \sqrt{\frac{\Gamma(1+p)}{\Gamma(1+n)}} \frac{\exp(-x/2)}{\Gamma(1+p-n)} x^{\frac{p-n}{2}} \Phi(-n, p-n+1; x); \tag{28}$$

and the subscripts of the Laguerre functions p_s, n_s $(s = 0,\, 1,\, 2)$ have the form

$$\begin{aligned} n_s &= \frac{1}{E}\left\{\gamma\left[l + \mu - \frac{s(3-s)}{4}\right] - k_0\alpha\right\} + \frac{(3s+1)(s-2)}{4} - \sqrt{\left[l + \mu - \frac{s(3-s)}{4}\right]^2 - \alpha^2}, \\ p_s &= \frac{1}{E}\left\{\gamma\left[l + \mu - \frac{s(3-s)}{4}\right] - k_0\alpha\right\} + \frac{(2-3s)(s-1)}{4} + \sqrt{\left[l + \mu - \frac{s(3-s)}{4}\right]^2 - \alpha^2}. \end{aligned}$$

For non-negative integer n_s, the functions (27) are square-integrable for $A \neq 0$ and $B = 0$, and the energy k_0 is quantized. In the case of the Dirac equation, we find

$$k_0 = \frac{1}{N^2 + \alpha^2}\left[\alpha\,\gamma\left(l + \mu - \frac{1}{2}\right) + N\sqrt{(N^2 + \alpha^2)(m^2 + k_3^2 + \gamma^2) - \gamma^2\left(l + \mu - \frac{1}{2}\right)^2}\right],$$

$$N = n + \sqrt{\left(l + \mu - \frac{1}{2}\right)^2 - \alpha^2}; \quad \left(l + \mu - \frac{1}{2}\right)^2 \geqslant \alpha^2; \quad n_1 = n - 1, \quad n_2 = n = 0,\, 1,\, 2,\, \dots\,,$$

whereas in the case of the Klein-Gordon equation we have

$$\begin{aligned} k_0 &= \frac{1}{\overline{N}^2 + \alpha^2}\left[\alpha\,\gamma\,(l + \mu) + \overline{N}\sqrt{(\overline{N}^2 + \alpha^2)(m^2 + k_3^2 + \gamma^2) - \gamma^2\,(l + \mu)^2}\right], \\ \overline{N} &= n + \frac{1}{2} + \sqrt{(l + \mu)^2 - \alpha^2}; \quad (l + \mu)^2 \geqslant \alpha^2; \quad n_0 = n. \end{aligned}$$

3.1.2 Subcase 2

This subcase is characterized by the conditions

$$f_0(r) = 0, \quad f_1(r) = f(r) \neq 0; \quad \mathbf{E} = 0, \quad \frac{e}{c\hbar}\mathbf{H} = f'(r)\mathbf{e}_\varphi + r^{-1}f_2'(r)\mathbf{k}\ .$$

Thus, we are dealing with a pure magnetic field. In such a field, the Dirac equation admits a spin integral of motion $T_2 = (\mathbf{\Sigma P})$, see [30]. Then we can impose an additional condition on the wave function,

$$(\mathbf{\Sigma P})\Psi_{D;l}(x^\nu) = \zeta\lambda_2\Psi_{D;l}(x^\nu), \quad \zeta = \pm 1, \quad \lambda_2 = \sqrt{k_0^2 - m^2}. \tag{29}$$

Eqs. (29) and (19) are consistent if we set

$$\begin{aligned}
\psi_1(r) &= \sqrt{k_0 + m}\,\overline{\varphi}_1(r), \quad \psi_2(r) = \sqrt{k_0 + m}\,\overline{\varphi}_2(r), \\
\psi_3(r) &= \zeta\sqrt{k_0 - m}\,\overline{\varphi}_1(r), \quad \psi_4(r) = \zeta\sqrt{k_0 - m}\,\overline{\varphi}_2(r),
\end{aligned}$$

in (19), where the functions $\overline{\varphi}_1(r)$ and $\overline{\varphi}_2(r)$ obey equations that are similar to that of (24),

$$\begin{aligned}
&[\zeta\lambda_2 + k_3 - f(r)]\overline{\varphi}_1(r) + \left[\frac{f_2(r) - l - \mu}{r} - \frac{d}{dr}\right]\overline{\varphi}_2(r) = 0, \\
&\left[\frac{f_2(r) - l - \mu + 1}{r} + \frac{d}{dr}\right]\overline{\varphi}_1(r) + [\zeta\lambda_2 - k_3 + f(r)]\overline{\varphi}_2(r) = 0.
\end{aligned} \tag{30}$$

Now, the Klein-Gordon equation (16) takes the form

$$\left\{\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - [k_3 - f(r)]^2 - \frac{[f_2(r) - l - \mu]^2}{r^2} + k_0^2 - m^2\right\}\psi_K(r) = 0. \tag{31}$$

Equations (30) and (31) have exact solutions (for nonzero $f(r)$) only for

$$f(r) = \frac{\alpha}{r}, \quad f_2(r) = \gamma r; \quad \alpha \neq 0,\, \gamma = \text{const}.$$

Such solutions have the form (they are similar to those in (26)

$$\begin{aligned}
&\overline{\varphi}_1(r) = \sqrt{q_{2(-)}\omega_2(\zeta)}\,v_1(x) - \zeta\sqrt{q_{2(+)}\omega_2(-\zeta)}\,v_2(x), \\
&\overline{\varphi}_2(r) = \sqrt{q_{2(+)}\omega_2(\zeta)}\,v_1(x) + \zeta\sqrt{q_{2(-)}\omega_2(-\zeta)}\,v_2(x), \quad \psi_K(r) = v_0(x); \\
&q_{2(\pm)} = \sqrt{\left(l + \mu - \frac{1}{2}\right)^2 + \alpha^2} \pm \left(l + \mu - \frac{1}{2}\right), \quad E = \sqrt{m^2 + k_3^2 + \gamma^2 - k_0^2}\,, \\
&\omega_2(\zeta) = \lambda_2\sqrt{\left(l + \mu - \frac{1}{2}\right)^2 + \alpha^2} + \zeta[k_3\left(l + \mu - \frac{1}{2}\right) - \gamma\alpha], \quad x = 2rE.
\end{aligned} \tag{32}$$

The functions $v_s(x)$, $(s = 0,\, 1,\, 2)$ are given by expressions (27), with the following indices of the Laguerre functions

$$\begin{aligned}
n_s &= \frac{1}{E}\left\{\gamma\left[l + \mu - \frac{s(3-s)}{4}\right] + k_3\alpha\right\} + \frac{(3s+1)(s-2)}{4} - \sqrt{\left[l + \mu - \frac{s(3-s)}{4}\right]^2 + \alpha^2}\,, \\
p_s &= \frac{1}{E}\left\{\gamma\left[l + \mu - \frac{s(3-s)}{4}\right] + k_3\alpha\right\} + \frac{(2-3s)(s-1)}{4} + \sqrt{\left[l + \mu - \frac{s(3-s)}{4}\right]^2 + \alpha^2}\,.
\end{aligned}$$

For non-negative integer n_s the functions (32) are square-integrable for $A \neq 0$ and $B = 0$, and the energy k_0 is quantized ($n_1 = n-1, \quad n_2 = n_0 = n = 0, 1, 2, ...$)

$$k_0^2 = m^2 + k_3^2 + \gamma^2 - \left[\frac{\gamma\left(l+\mu-\frac{\tau}{2}\right)+k_3\alpha}{n+\frac{1-\tau}{2}+\sqrt{\left(l+\mu-\frac{\tau}{2}\right)^2+\alpha^2}} \right]^2 , \tag{33}$$

where $\tau = 1$ for the Dirac equation and $\tau = 0$ for the Klein-Gordon equation.

3.1.3 Subcase 3

This subcase is characterized by the conditions

$$\begin{aligned}
&f_0(r) = f(r) \neq 0, \quad f_1(r) = \epsilon f(r), \quad \epsilon^2 = 1; \\
&\frac{e}{c\hbar}\mathbf{E} = f'(r)(\epsilon\mathbf{k} - \mathbf{e}_r), \quad \frac{e}{c\hbar}\mathbf{H} = \epsilon f'(r)\mathbf{e}_\varphi + r^{-1}f_2'(r)\mathbf{k} \ .
\end{aligned}$$

In this subcase, the fields have the following properties

$$\mathbf{H} = [\mathbf{n}\times\mathbf{E}] + \mathbf{n}(\mathbf{nH}), \quad \mathbf{E} = -[\mathbf{n}\times\mathbf{H}] + \mathbf{n}(\mathbf{nE}), \quad \mathbf{n} = -\epsilon\mathbf{k}. \tag{34}$$

If eqs. (34) hold, then the bispinor $\overline{\Phi}(r)$ in eq. (19) can be represented as

$$\begin{aligned}
&\overline{\Phi}(r) = \begin{pmatrix} \left[k_0 - \epsilon k_3 + m - \epsilon\hat{Q}\sigma_3\right] V(r) \\ \left[\epsilon(k_0 - \epsilon k_3 - m)\sigma_3 - \hat{Q}\right] V(r) \end{pmatrix}, \quad V(r) = \begin{pmatrix} v_1(r) \\ v_2(r) \end{pmatrix}, \\
&\hat{Q} = \sigma_1 \frac{f_2(r) - l - \mu + 1/2}{r} - i\sigma_2\left(\frac{d}{dr} + \frac{1}{2r}\right),
\end{aligned} \tag{35}$$

see [30]. The functions $v_1(r)$, $v_2(r)$ and $\psi_K(r) = v_0(r)$ obey the equations

$$\begin{aligned}
&\left\{\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - 2(k_0 - \epsilon k_3)f(r) - \frac{[f_2(r) - l - \mu + \tau_s]^2}{r^2} + \delta_s\frac{f_2'(r)}{r} + k_0^2 - k_3^2 - m^2\right\} v_s(r) = 0, \\
&s = 0,\ 1,\ 2; \quad \tau_s = s(2-s); \quad \delta_s = \frac{s(5-3s)}{2}.
\end{aligned} \tag{36}$$

Equations (36) have exact solutions for the following two types of the functions $f(r)$ and $f_2(r)$:

a)

$$\begin{aligned}
&f_2(r) = \gamma r, \quad f(r) = \frac{\alpha}{r} + \frac{\beta}{r^2}; \\
&x = 2rE, \quad \alpha, \beta, \gamma = const; \\
&E = \sqrt{m^2 + k_3^2 + \gamma^2 - k_0^2};
\end{aligned} \tag{37}$$

b)

$$\begin{aligned}
&f_2(r) = \gamma r^2, \quad f(r) = \alpha r^2 + \frac{\beta}{r^2}; \\
&x = r^2 E_0, \quad \alpha, \beta, \gamma = \mathrm{const}; \\
&E_0 = \sqrt{\gamma^2 + 2\alpha(k_0 - \epsilon k_3)}.
\end{aligned} \tag{38}$$

As before, the functions $v_s(x)$, $(s = 0,\, 1,\, 2)$ are given by expressions (27), where the subindices of the Laguerre functions n_s, p_s have the form

a)

$$\begin{aligned} p_s &= \frac{b_s}{E} - \frac{1}{2} + a_s, \quad n_s = \frac{b_s}{E} - \frac{1}{2} - a_s, \\ b_s &= \gamma(l + \mu - \tau_s + \frac{\delta_s}{2}) - \alpha(k_0 - \epsilon k_3), \\ a_s &= \sqrt{(l + \mu - \tau_s)^2 + 2\beta(k_0 - \epsilon k_3)}; \end{aligned} \tag{39}$$

b)

$$\begin{aligned} p_s &= \frac{\tilde{b}_s}{4E_0} - \frac{1}{2} + \frac{a_s}{2}, \quad n_s = \frac{\tilde{b}_s}{4E_0} - \frac{1}{2} - \frac{a_s}{2}, \\ \tilde{b}_s &= 2\gamma(l + \mu - \tau_s + \delta_s) + k_0^2 - k_3^2 - m^2, \\ a_s &= \sqrt{(l + \mu - \tau_s)^2 + 2\beta(k_0 - \epsilon k_3)}. \end{aligned} \tag{40}$$

For non-negative integer n_s, the functions (37) are square-integrable for $A \neq 0$ and $B = 0$, and the energy k_0 is quantized $(n_1 = n - 1,\;\; n_2 = n_0 = n = 0,\, 1,\, 2,\, ...)$. In the general case an explicit expression for the energy k_0 has not been obtained previously. Such expressions can be written for fields of the type a) as a root of an algebraic equation of power six, and for fields of the type b) as a root of an algebraic equation of power eight.

3.2 Case II

In this case, the functions f_0 and f_1 depend on z and x_0 and do not depend on r ($f_{0,1} = f_{0,1}(z,\, x_0)$).

As follows from (11), additional fields have the form

$$\frac{e}{c\hbar}\mathbf{E} = F\mathbf{k}, \quad \frac{e}{c\hbar}\mathbf{H} = G\mathbf{k}; \quad F = \partial_0 f_1(z,\, x_0) - \partial_z f_0(z, x_0), \quad G = f_2'(r) r^{-1}\,. \tag{41}$$

According to a classification presented in [30], these are longitudinal electromagnetic fields, the electric and magnetic fields are parallel to the axis z, and in addition $\mathbf{H} = \mathbf{H}\,(r)$ and $\mathbf{E} = \mathbf{E}\,(z, x_0)$.

As was demonstrated in [30], for such fields, solutions of the Klein-Gordon equation (16) can be found in the form

$$\psi_K(r, z, x_0) = w_0(z,\, x_0) v_0(r). \tag{42}$$

As to solutions of the Dirac equation (18), we represent the bispinor $\overline{\Phi}(r,\, z,\, x_0)$ in the form

$$\overline{\Phi}(r,\, z,\, x_0) = \begin{pmatrix} \left[m + \pi_0 + \pi_3 + \hat{Q}\sigma_3\right] w_1(z,\, x_0) V(r) \\ \left[(m - \pi_0 - \pi_3)\sigma_3 - \hat{Q}\right] w_1(z,\, x_0) V(r) \end{pmatrix}, \tag{43}$$

where

$$\begin{aligned} &\pi_0 = i\partial_0 - f_0(z,\, x_0), \;\; \pi_3 = i\partial_z - f_1(z,\, x_0), \\ &V(r) = \begin{pmatrix} v_1(r) \\ v_2(r) \end{pmatrix}, \quad \hat{Q} = \sigma_1 \frac{f_2(r) - l - \mu + 1/2}{r} - i\sigma_2 \left(\frac{d}{dr} + \frac{1}{2r}\right). \end{aligned}$$

The functions $v_s(r)$, $s = 0, 1, 2$, and functions $w_\nu(z, x_0)$, $\nu = 0, 1$, in eqs. (42) and (43) obey the equations

$$\left\{\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{[f_2(r) - l - \mu + s(2-s)]^2}{r^2} + \frac{s(5-3s)}{2}\frac{f_2'(r)}{r} + k_\perp^2\right\} v_s(r) = 0, \qquad (44)$$

$$\left\{\pi_0^2 - m^2 - \pi_3^2 - k_\perp^2 + i\nu[\partial_z f_0(z, x_0) - \partial_0 f_1(z, x_0)]\right\} w_\nu(z, x_0) = 0, \quad k_\perp^2 = \text{const}. \quad (45)$$

Exact solutions of eq. (44) (for $f_2(r) \neq 0$) are known in the two cases (21) and for $f_2(r) = \gamma r$, ($\gamma = \text{const}$) have the form (27) if we set **there** $x = 2r\sqrt{\gamma^2 - k_\perp^2}$ and

$$\begin{aligned} p_s &= \frac{\gamma}{\sqrt{\gamma^2 - k_\perp^2}}\left[l + \mu + \frac{s(s-3)}{4}\right] - \frac{1}{2} + |l + \mu - s(2-s)|, \\ n_s &= \frac{\gamma}{\sqrt{\gamma^2 - k_\perp^2}}\left[l + \mu + \frac{s(s-3)}{4}\right] - \frac{1}{2} - |l + \mu - s(2-s)|. \end{aligned}$$

For $\gamma^2 > k_\perp^2$, $\gamma(l+\mu-\tau) \geqslant 0$ ($\tau = 1/2$ for the Dirac equation and $\tau = 0$ for the Klein-Gordon equation) the quantity $k_\perp^2$ is quantized and the wave functions (27) are square-integrable for $A \neq 0$, $B = 0$,

$$k_\perp^2 = \gamma^2\left[1 - \frac{(l + \mu - \tau)^2}{(n + \frac{1}{2} + |l + \mu|)^2}\right], \; n_0 = n_2 = n = 0, 1, 2, ..., \; n_1 = n - \frac{l}{|l|}. \qquad (46)$$

The second case $f_2(r) = \gamma r^2$ corresponds to the so-called magnetic-solenoid field (the exactly solvable additional field is a constant uniform magnetic fields parallel to the AB solenoid) ($G = 2\gamma$ in eq. (41)). For such a field, exact solutions have been studied in detail in numerous works, see e.g. [12, 16, 17, 19, 20], and we do not repeat these results here.

Let us consider exact solutions of the equation (45). A wide class of exact solutions can be found if both functions $f_{0,1}(z, x_0)$ depend on one variable only: $f_{0,1} = f_{0,1}(x)$, where either $x = z$ (in such a case, without loss of generality, we can set $f_0(z) = f(x) \neq 0, f_1(z) = 0$); or $x = x_0$ (and here also without loss of generality, we can set $f_0(x_0) = 0$, $f_1(x_0) = f(x) \neq 0$). If $x = z$, then we can select $w_\nu(x, x_0)$ as an eigenfunction of $i\partial_0$, in this case we find

$$\begin{aligned} &i\partial_0 w_\nu(x, x_0) = k_0 w_\nu(x, x_0) \Rightarrow w_\nu(x, x_0) = \exp(-ik_0 x_0) w_\nu(x), \\ &\left\{\frac{d^2}{dx^2} + [k_0 - f(x)]^2 - m^2 - k_\perp^2 + i\nu f'(x)\right\} w_\nu(x) = 0. \end{aligned} \qquad (47)$$

If $x = x_0$, then $w_\nu(z, x)$ can be selected as an eigenfunction of $i\partial_z$. In this case we find

$$\begin{aligned} &i\partial_z w_\nu(z, x) = k_3 w_\nu(z, x) \Rightarrow w_\nu(z, x) = \exp(-ik_3 z) w_\nu(x), \\ &\left\{\frac{d^2}{dx^2} + [k_3 - f(x)]^2 + m^2 + k_\perp^2 + i\nu f'(x)\right\} w_\nu(x) = 0. \end{aligned} \qquad (48)$$

Equations (47) and (48) are one-dimensional Schrödinger equations. Their exact solutions exist for the following functions $f(x)$:

$$\begin{aligned} f(x) &= \alpha\, x; \;\; f(x) = \alpha/x; \;\; f(x) = \alpha \exp(\beta x); \\ f(x) &= \alpha\, \tan(\beta x); \;\; f(x) = \alpha \tanh(\beta x); \;\; f(x) = \alpha \coth(\beta x), \end{aligned} \qquad (49)$$

where $\alpha, \beta = \text{const}$. Such solutions are well-known, see e.g. [30].

In two cases when the function $f_{0,1}(z,x_0)$ depends essentially on both arguments z, x_0, one can find exact solutions of eq. (45).

1) Let us set $f_0(z,x_0) = f_1(z,x_0) = \frac{1}{2}f(\xi)$, $\xi = x_0 - z$; in this case $F = f'(\xi)$ in eq. (41). Then $w_\nu(z,x_0)$ can be selected as an eigenfunction of $i(\partial_0 + \partial_z)$ with the eigenvalue λ, which implies

$$\begin{aligned} & w_\nu(z,x_0) = [\lambda - f(\xi)]^{-\frac{1+\nu}{2}} \exp(iS), \\ & S = -\frac{1}{2}\left[\lambda(x_0+z) + (m^2 + k_\perp^2)\int \frac{d\xi}{\lambda - f(\xi)}\right]. \end{aligned} \tag{50}$$

2) Let us set $f_0(z,\, x_0) = -\frac{z}{\bar{\xi}}f(\bar{\xi})$, $\;f_1(z,x_0) = \frac{x_0}{\bar{\xi}}f(\bar{\xi})$; $\;\bar{\xi} = x_0^2 - z^2$; in this case $F = 2f'(\bar{\xi})$ in eq. (41). Then $w_\nu(z,x_0)$ can be selected as an eigenfunction of the operator $\hat{q} = i(z\partial_0 + x_0\partial_z)$ (this operator is an integral of motion for the Klein-Gordon and Dirac equations) with the eigenvalue λ, which implies

$$\hat{q}w_\nu(z,x_0) = \lambda w_\nu(z,x_0) \Rightarrow w_\nu(z,x_0) = \left(\frac{x_0 - z}{x_0 + z}\right)^{i\frac{\lambda}{2}} w_\nu(\bar{\xi}). \tag{51}$$

Substituting (51) into (45), we find an equation for the function $w_\nu(\bar{\xi})$,

$$4\bar{\xi}^2 w_\nu''(\bar{\xi}) + 4\bar{\xi}w_\nu'(\bar{\xi}) + \left\{[\lambda - f(\bar{\xi})]^2 + \bar{\xi}[m^2 + k_\perp^2 + 2i\nu f'(\bar{\xi})]\right\} w_\nu(\bar{\xi}) = 0. \tag{52}$$

Equation (52) can be solved exactly in two cases:

a)

$$f(\bar{\xi}) = \alpha\bar{\xi}, \quad \alpha = \text{const},$$

which corresponds to a constant and uniform electric field;

b)

$$f(\bar{\xi}) = \alpha\sqrt{|\bar{\xi}|}, \quad \alpha = \text{const}.$$

In these cases solutions have the form (27),

$$w_\nu(\bar{\xi}) = AI_{p,n}(x) + BI_{n,p}(x), \quad A,\, B = \text{const}, \tag{53}$$

where one has to set respectively

a)

$$f(\bar{\xi}) = \alpha\bar{\xi},\ \alpha = \text{const}, \quad x = -i\alpha\bar{\xi}, \quad p = \frac{i(m^2 + k_\perp^2) - 2\alpha(1+\nu)}{4\alpha},\ \ n = p - i\lambda;$$

and

b)

$$\begin{aligned} & f(\bar{\xi}) = \alpha\sqrt{|\bar{\xi}|}, \quad \alpha = \text{const}, \quad x = 2i\sqrt{\alpha^2|\bar{\xi}| + \bar{\xi}(m^2 + k_\perp^2)}, \\ & p = \frac{\alpha(\nu + 2i\lambda)}{\sqrt{\alpha^2 + \varepsilon(m^2 + k_\perp^2)}} - 1/2 + i\lambda, \quad n = p - 2i\lambda, \quad \varepsilon = \bar{\xi}/|\bar{\xi}|\,. \end{aligned}$$

Thus, with the above consideration, all the exactly solvable additional electromagnetic fields in the cylindric coordinates have been exhausted.

Schrödinger equation

Let us consider the Schrödinger equation (15). Exact solutions of this equation are known also only for two types of function $f_2(r)$.

Let us suppose that additional fields can depend on the constant m (on the particle mass). Then:

a) For $f_2(r) = \gamma r$ one can find exact solutions for

$$f_0(r) = \frac{\alpha}{r} + \frac{\delta}{r^2} + \frac{2\lambda}{r^3}\left(\beta - \frac{m\lambda}{r}\right), \quad f_1(r) = \frac{\beta}{r} - \frac{2m\lambda}{r^2}; \quad \alpha,\ \beta,\ \delta,\ \lambda = \text{const}. \tag{54}$$

The general solution of the Schrödinger equation (15) can be expressed via the Laguerre functions and has the form

$$\psi_S(r) = AI_{p,n}(x) + BI_{n,p}(x), \quad x = 2r\sqrt{E}, \quad p = \frac{b}{\sqrt{E}} - \frac{1}{2} + \sqrt{a}, \quad n = \frac{b}{\sqrt{E}} - \frac{1}{2} - \sqrt{a};$$
$$a = (l+\mu)^2 + \beta^2 + 2m\delta + 4m\lambda k_3, \quad b = \gamma(l+\mu) + \beta k_3 - \alpha m, \quad E = \gamma^2 + k_3^2 - 2mk_0, \tag{55}$$

where A and B are arbitrary constants. For $a \geqslant 0$ and non-negative integer $n = 0,\ 1,\ 2,\ ...,$ the functions (20) and (21) are square-integrable at $B = 0$, and the non-relativistic particle energy k_0 is quantized,

$$\psi_S(r) = A\exp(-x/2)x^{\sqrt{a}}L_n^{2\sqrt{a}}(x), \quad k_0 = \frac{1}{2m}\left[\gamma^2 + k_3^2 - \frac{4b^2}{(2n+1+2\sqrt{a})^2}\right], \tag{56}$$

where $L_n^\alpha(x)$ are Laguerre polynomials (see [31], eq. 8.970.1).

b) $f_2(r) = \gamma r^2$ one can find exact solutions for $\alpha,\ \beta,\ \gamma,\ \delta,\ \lambda = \text{const}$, and

$$f_0(r) = \alpha r^2 + \frac{\beta}{r^2} - 2m\left(\frac{\lambda^2}{r^4} + \delta^2 r^4\right), \quad f_1(r) = -2m\left(\frac{\lambda}{r^2} + \delta r^2\right). \tag{57}$$

As in the previous case, the general solution of the Schrödinger equation (15) can be expressed via the Laguerre functions and has the form

$$\psi_S(r) = AI_{p,n}(x) + BI_{n,p}(x),$$

where A and B are arbitrary constants and

$$\begin{aligned} x &= \sqrt{b}\,r^2, \quad p = \frac{E}{4\sqrt{b}} - \frac{1}{2} + \frac{\sqrt{a}}{2}, \quad n = \frac{E}{4\sqrt{b}} - \frac{1}{2} - \frac{\sqrt{a}}{2}, \\ a &= (l+\mu)^2 + 2\beta m + 4m\lambda k_3, \quad b = \gamma^2 + 2\alpha m + 4m\delta k_3, \\ E &= 2mk_0 + 2\gamma(l+\mu) - k_3^2 - 8\delta\lambda m^2. \end{aligned} \tag{58}$$

For $a \geqslant 0$, $b > 0$ and non-negative integer n, the function $\psi_S(r)$ is square-integrable and the non-relativistic particle energy k_0 is quantized,

$$k_0 = \frac{1}{2m}\left[2\sqrt{b}(2n+1+\sqrt{a}) + k_3^2 - 2\gamma(l+\mu) + 8\delta\lambda m^2\right],$$
$$\psi_S(r) = A\exp(-x/2)x^{\frac{\sqrt{a}}{2}}L_n^{\sqrt{a}}(x).$$

If additional fields do not depend on m, then one has to set $\lambda = 0$ in eqs. (54) and (55), and $\delta = \lambda = 0$ in eqs. (57) and (58).

For the fields of case II, solutions of the Schrödinger equation (15) can be written as

$$\psi_S(r, z, x_0) = w_S(z, x_0) v_0(r), \tag{59}$$

where the function $v_0(r)$ is a solution of eq. (44) (for $s = 0$). Exact solutions are known only for the functions (21). The function $w_S(z,\, x_0)$ is a solution of the equation

$$\left\{2m[i\partial_0 - f_0(z, x_0)] - k_\perp^2 - [i\partial_z - f_1(z, x_0)]^2\right\} w_S(z, x_0) = 0. \tag{60}$$

If $f_0(z,\, x_0) = f(z)$ and $f_0(z,\, x_0) = 0$, the function $w_S(z,\, x_0)$ can be selected as an eigenfunction for the operator $i\partial_0$,

$$i\partial_0 w_S(z,\, x_0) = k_0 w_S(z, x_0) \Rightarrow w_S(z, x_0) = \exp(-ik_0 x_0) w_S(z). \tag{61}$$

Taking into account (60), we obtain for $w_S(z)$ the one-dimensional Schrödinger equation following equation

$$\left\{2m[k_0 - f(z)] - k_\perp^2 + \frac{d^2}{dz^2}\right\} w_S(z) = 0 \tag{62}$$

already discussed above.

If $f_0(z,\, x_0) = 0$ and $f_1(z,\, x_0) = f(x_0)$ (which correspond to a uniform electric field that depends on time), the function $w_S(z,\, x_0)$ can be selected as an eigenfunction for the operator $i\partial_z$,

$$i\partial_z w_S(z,\, x_0) = k_3 w_S(z,\, x_0) \Rightarrow w_S(z,\, x_0) = \exp(-ik_3 z) w_S(x_0). \tag{63}$$

Taking into account (60), we obtain for $w_S(x_0)$ the equation

$$\left\{2im\partial_0 - k_\perp^2 - [k_3 - f(x_0)]^2\right\} w_S(x_0) = 0. \tag{64}$$

Its exact solution can be found for an arbitrary $f(x_0)$,

$$w_S(x_0) = \exp[-iS(x_0)], \quad S(x_0) = \frac{k_\perp^2}{2m} x_0 + \int \frac{[k_3 - f(x_0)]^2}{2m}\, dx_0. \tag{65}$$

For arbitrary functions $f_{0,1}(z,\, x_0)$, equation (60) can be reduced to the one-dimensional Schrödinger equation

$$i\partial_0 \psi(z,\, x_0) = \mathcal{H}\psi(z,\, x_0), \quad \mathcal{H} = -\frac{1}{2m}\frac{d^2}{dz^2} + \mathcal{V}(z,\, x_0), \tag{66}$$
$$\mathcal{V}(z,\, x_0) = f_0(z,\, x_0) - \int [\partial_0 f_1(z,\, x_0)]\, dz$$

with the help of the substitution

$$w_S(z,\, x_0) = \exp[-i\varphi(z, x_0)]\psi(z,\, x_0), \quad \varphi(z, x_0) = \frac{k_\perp^2}{2m} x_0 + \int f_1(z,\, x_0) dz. \tag{67}$$

We note that the potential in such an equation depends on time.

We believe that there exist new kinds of functions $f_{0,1}(z,\, x_0)$ for which exact solutions can be found. The method of constructing such functions and corresponding solutions is developed in [32, 33].

Thus, with the above consideration, all the exactly solvable additional electromagnetic fields in the case of the Schrödinger equation in the cylindric coordinates have been exhausted.

4 Structure of additional electromagnetic fields. Spherical coordinates

In the spherical coordinates r, θ, φ,

$$x = r\sin\theta\cos\varphi, \quad y = r\sin\theta\sin\varphi, \quad z = r\cos\theta \tag{68}$$

potentials of the AB field (2) have the form

$$A_\nu^{(0)} = \left(A_0^{(0)}, -\mathbf{A}^{(0)}\right), \; \mathbf{A}^{(0)} = \frac{\Phi}{2\pi r\sin\theta}\mathbf{e}_\varphi, \; \mathbf{e}_\varphi = -\mathbf{i}\sin\varphi + \mathbf{j}\cos\varphi\,. \tag{69}$$

Potentials A_ν of additional fields we write as

$$\frac{e}{c\hbar}A_0 = f_0(r), \; \frac{e}{c\hbar}\mathbf{A} = \frac{f_1(\cos\theta)}{r\sin\theta}\mathbf{e}_\varphi, \tag{70}$$

where f_k $(k = 0, 1)$ are arbitrary functions of their arguments. Thus, the potentials of the AB field (2) can be considered as a particular case of the additional field (70) for $f_0(r) = 0$ and $f_1(\cos\theta) = \Phi/2\pi = \text{const}$.

Electric and magnetic field that corresponds to (70) have the form

$$\frac{e}{c\hbar}\mathbf{E} = -f_0'(r)\mathbf{e}_r, \; \frac{e}{c\hbar}\mathbf{H} = -\frac{f_1'(\cos\theta)}{r^2}\mathbf{e}_r, \quad \mathbf{e}_r = \sin\theta(\mathbf{i}\cos\varphi + \mathbf{j}\sin\varphi) + \mathbf{k}\cos\theta\,. \tag{71}$$

A complete procedure of variable separation and finding complete sets of integrals of motion for the Schrödinger, Klein-Gordon and Dirac equations with potentials (70) is presented in detail in [30].

One can find exact solutions of the Klein-Gordon and Dirac equations for

$$f_1(\cos\theta) = \alpha\cos\theta + \beta, \quad f_0(r) = \frac{\gamma}{r}, \quad \alpha, \beta, \gamma = \text{const}. \tag{72}$$

In this case the complete external electromagnetic field is a combination of the AB field with the Coulomb field and the field of a magnetic monopole. The corresponding exact solutions were studied in the works [23, 24, 25, 26, 27, 13].

As to the Schrödinger equation, its solutions can be found for a more general potentials

$$f_1(\cos\theta) = \alpha\cos\theta + \beta, \quad f_0(r) = \frac{\gamma}{r} + \frac{\delta}{r^2}, \quad \alpha, \beta, \gamma, \delta = \text{const}. \tag{73}$$

The Schrödinger equation with potentials (73) can be trivially identified with the Klein-Gordon equation with the potentials (72) and, therefore, does not need a separate consideration.

5 Conclusion

Exact solutions of quantum mechanical wave equations for a combination of Aharonov-Bohm field with additional external electromagnetic fields has an undoubted physical interest. The additional fields can emphasize or even reinforce some specific manifestations of the Aharonov–Bohm effect. As was already said, exact solutions in the case when the additional field is a constant and uniform magnetic field, revealed non-trivial manifestations of the Aharonov-Bohm effect in the synchrotron radiation.

Only some of additional fields and the corresponding exact solutions, which are considered above, have been sufficiently studied until now. Namely, a constant and uniform magnetic field that is parallel to the Aharonov-Bohm solenoid, a static spherically symmetrical electric field (in particular Coulomb field), and a field of a magnetic monopole.

In this work we have demonstrated that aside from these known cases, there are broad classes of additional external fields, for which exact solutions exist for both relativistic and non relativistic wave equations. Among these new additional fields we have physically interesting electric fields acting during a finite time, or localized in a finite region of space (see the potentials in (49). The corresponding exact solutions can be used in non perturbative calculations of different processes in QED with unstable vacuum (see. [34]) and, therefore, to study the Aharonov-Bohm effect in such processes. There are additional time-dependent uniform and isotropic electric fields that allow exact solutions of the Schrödinger equation. It should be also noted that in the relativistic case these are additional electric fields propagating along the Aharonov-Bohm solenoid with arbitrary electric pulse shape (50).

Obtained results give a possibility to investigate precisely manifestations of the Aharonov-Bohm effect in the background of a wide class of additional electromagnetic fields.

Acknowledgments

The work of VGB was partially supported by the Federal Targeted Program "Scientific and scientific - pedagogical personnel of innovative Russia", contract 02.740.11.0238; N. P789; DMG acknowledges the permanent support of FAPESP and CNPq.

References

[1] Y. Aharonov and D. Bohm, *Significance of electromagnetic potentials in quantum theory,* Phys. Rev. **115** (1959) 485-491

[2] W. Ehrenberg and R.E. Siday, *The Refractive Index in Electron Optics and the Principles of Dynamics*, Proc. Phys. Soc. Lond., B **62** (1949) 8-212

[3] T.T. Wu and C.N. Yang, *Concept of nonintegrable phase factors and global formulation of gauge fields*, Phys. Rev. D **12** (1975) 3845-3857

[4] U. Percoco, V.M. Villalba, *Aharonov - Bohm Effect for a Relativistic Dirac Electron*, Phys. Lett. A **140**(3) (1989) 105-107

[5] C.R. Hagen, *Aharonov-Bohm Scattering of Particles with Spin*, Phys. Rev. Letters, **64** **(**1990) 503-506; *Spin dependence of the Aharonov–Bohm effect*, Int. J. Mod. Phys. A **6** (1991) 3119-3149

[6] C.R. Hagen, *Effects of nongauge potentials on the spin-1/2 Aharonov-Bohm problem*, Phys. Rev. **D 48** (1993) 5935-5939

[7] S.A. Voropaev, D.V. Galtsov, and D.A. Spasov, *Bound states for fermions in the gauge Aharonov-Bohm field*, Phys. Lett. B **267** (1991) 91-94

[8] Y. Nambu, *The Aharonov-Bohm problem revisited*, Nucl. Phys. B **579** (2000) 590-616; M. Hirokawa and O. Ogurisu, *Ground state of a spin-1/2 charged particle in a two-dimensional magnetic field*, J. Math. Phys., **42** (2001) 3334-3343

[9] F.A.B. Coutinho, Y. Nogami, and J.F. Perez, *Self-adjoint extensions of the Hamiltonian for a charged-particle in the presence of a thread of magnetic-flux*, Phys. Rev. A **46** (1992) 6052-6055; *Self-adjoint extensions of the Hamiltonian for a charged spin-1/2 particle in the Aharonov-Bohm field*, Journ. Phys. A **27** (1994) 6539-6550

[10] F.A.B. Coutinho and J.F. Perez, *Helicity conservation in the Aharonov-Bohm scattering of Dirac Particles*, Phys. Rev. D **49** (1994) 2092-2097

[11] H. Falomir and P.A.G. Pisani, *Hamiltonian self-adjoint extensions for (2+1)-dimensional Dirac particles,* Journ. Phys. A **34** (2001) 4143-4154

[12] R.R. Lewis, *Aharonov-Bohm effect for trapped ions,* Phys. Rev. A **28** (1983) 1228-1236

[13] V.G. Bagrov, D.M. Gitman, and V.B. Tlyachev, *Solutions of relativistic wave equations in superpositions of Aharonov-Bohm, magnetic, and electric fields*, J. Math. Phys. **42** (2001) 1933-1959

[14] P. Exner, P. Šťovíček, and P. Vytřas, *Generalized boundary conditions for the Aharonov-Bohm effect combined with a homogeneous magnetic field*, J. Math. Phys., **43** (2002) 2151-2168

[15] S.P. Gavrilov, D.M. Gitman, A.A. Smirnov, and B.L. Voronov, *Dirac fermions in a magnetic-solenoid field,* hep-th/0308093; "Focus on Mathematical Physics Research" Ed. by Charles V. Benton (Nova Science Publishers, New York, 2004) 131-168

[16] V.G. Bagrov, S.P. Gavrilov, D.M. Gitman, and D.P. Meira Filho, *Coherent states of non-relativistic electron in magnetic–solenoid field,* Journ. Physics. A**43** (2010) 3540169 (10 pages)

[17] V.G. Bagrov, S.P. Gavrilov, D.M. Gitman, and D.P. Meira Filho, *Coherent and semiclassical states in magnetic field in the presence of the Aharonov-Bohm solenoid*, J. Phys. A: Math. Theor. **44** (2011) 055301 (35 pages).

[18] D.M. Gitman, I.V. Tyutin, A. Smirnov, and B.L. Voronov, *Self-adjoint Schrödinger and Dirac operators with Aharonov-Bohm and magnetic-solenoid fields*, arXiv:0911.0946 [quant-ph]

[19] V.G. Bagrov, D.M. Gitman, A. Levin, and V.B. Tlyachev, *Impact of Aharonov-Bohm Solenoid on Particle Radiation in Magnetic Field*, Mod. Phys. Lett. A **16** (2001) 1171-1179

[20] V.G. Bagrov, D.M. Gitman, A. Levin, and V.B. Tlyachev, *Aharonov-Bohm effect in cyclotron and synchrotron radiations*, Nucl. Phys. B **605** (2001) 425-454

[21] V.G. Bagrov, D.M. Gitman, and V.B. Tlyachev, *l-dependence of particle radiation in magnetic-solenoid field and Aharonov-Bohm effect*, Int. Journ. Mod. Phys. A**17** (2002) 1045-1048

[22] O. Lisovyy, *Aharonov-Bohm effect on the Poincare disc*, J. Math. Phys. **48** (2007) 052112-17

[23] V.M. Villalba, *Exact solutions of the Dirac equation for a Coulomb and scalar potential in the presence of an Aharonov-Bohm and magnetic monopole fields*, J. Math. Phys. **36** (1995) 3332-3344

[24] Le Van Hoang, Ly Xuan Hai, L. I. Komarov, T.S. Romanova, *Relativistic Analogy of the Aharonov – Bohm Effect in the Presence of Coulomb Field and Magnetic Charge,* Journal of Physics. A**25 (**25**)** (1992) 6461-6469

[25] V.R. Khalilov, *Relativistic Aharonov – Bohm Effect in the Presence of Planar Coulomb Potentials*, Phys. Rev. A **71**(1) (2005) 012105 (6 pages)

[26] A.D. Alhaidari, *The Three–Dimensional Dirac Oscillator in the Presence of Aharonov–Bohm and Magnetic Monopole Potentials*, Foundations of Physics Letters **18**(7) (2005) 651-664

[27] F.A.B. Coutinho and J.F. Perez, *Boundary-conditions in the Aharonov-Bohm scattering of Dirac particles and the effect of Coulomb interaction*, Phys. Rev. D **48** (1993) 932-939

[28] J. Audretsch, V. Skarzinsky, and B. Voronov, *Elastic scattering and bound states in the Aharonov-Bohm potential superinposed by an attractive ρ^{-2} potential*, Journ. Phys. A **34** (2001) 235-250

[29] V.G. Bagrov, M.C. Baldiotti, D.M. Gitman, and I.V. Shirokov, *New solutions of relativistic wave equations in magnetic field and longitudinal fields*, J. Math. Phys. **43** (2002) 2284-2295

[30] V.G. Bagrov and D.M. Gitman, *Exact Solutions of Relativistic Wave Equations*, Math. its Appl., Sov. ser., Vol **39** (Kluwer, Dordrecht 1990)

[31] I.S. Gradshtein and I.W. Ryzhik, *Table of Integrals, Series, and Products* (Academic Press, New York 1994)

[32] V.N. Shapovalov, N.B. Sukhomlin, *Separation of Variables in the Nonstationary Schrödinger Equation*, Soviet Phys. Journ. **17**(12) (1974) 1718-1722

[33] V.G. Bagrov, A.V. Shapovalov, I.V. Shirokov, *Generation on Exactly Solvable Potentials for the Nonstationary Schrödinger Equation*, Soviet Theor. Math. Phys. **87**(3) (1991) 426-433

[34] E.S.Fradkin, D.M. Gitman, and Sh.M. Shvartsman, *Quantum Electrodynamics with Unstable Vacuum* (Springer-Verlag, Berlin, 1991)